MODULE 3: MULTIPLYING AND FACTORING WHOLE NUMBERS

In Module 2 we saw the importance of being able to use multiples of powers of ten. There are times when it is equally as important to use multiples of numbers which aren't powers of ten.

Example 1

At a price of $7 each, what is the cost of 3 handkerchiefs?

Answer: $21

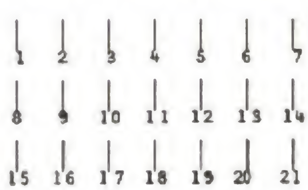
We want the sum of three 7's; that is, 7 + 7 + 7. In vertical form we have:

Don't confuse the sum of 3 sevens with the sum of 3 and 7.

$$\begin{array}{r} 7 \\ +\ \ 7 \\ \hline 14 \\ +\ \ 7 \\ \hline 21 \end{array}$$

The fact that 7 + 7 + 7 = 21 also tells us that $7 + $7 + $7 = $21

A more pictorial way of doing Example 1 would have been to use tally marks. We could have let one tally mark stand for $1. Then the cost of each handkerchief would be represented by 7 tally marks. We'd then represent the cost of 3 handkerchiefs by writing 3 rows, each with 7 tally marks. For example:



Looking at the last tally mark in each row, we see that if we count by 7's we get 7, 14, 21, . . .

Because 21 is the sum of three 7's, we call 21 the <u>3rd multiple</u> of 7.

That is, the 3rd multiple o 7 is the third number we come to when we "count by 7's"

Let's make sure you understand the new vocabulary.

Example 2

What is the 4th multiple of 9?

Answer: 36

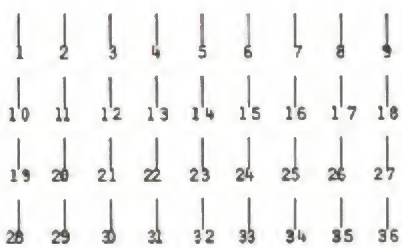
The 4th multiple of 9 is the sum of four 9's. In vertical form we have:

The sum of four 9's is written as 9 + 9 + 9 + 9

$$\begin{array}{rl} & 9 \text{ (1st multiple of 9)} \\ + & \underline{9} \\ & 18 \text{ (2nd multiple of 9)} \\ + & \underline{9} \\ & 27 \text{ (3rd multiple of 9)} \\ + & \underline{9} \\ & 36 \text{ (4th multiple of 9)} \end{array}$$

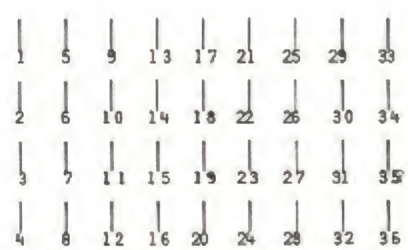
Module 2 is a prequisite for this module. That is, to find the 4th multiple of 9 we have to be able to find the sum of four 9's.

The tally mark interpretation yields a rather interesting result. For example when we write our 4 rows each with 9 tally marks, we can either count along the rows first or along the columns first. That is we could write:



Looking at the last tally in each row, we see that we count by 9's to get: 9, 18, 27, 36, ...as the multiples of 9.

or we could write:



Looking at the bottom row we see that the multiples of 4 are 4, 8, 12, 16, 20, 24, 28, 32, 36, . . .

By counting the same number of tally marks in two different orders, we see that the 4th multiple of 9 is the same number as the 9th multiple of 4. This result is often useful.

Without tally marks this result is not so obvious. That is, 9 + 9 + 9 + 9 doesn't look the same as 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4.

Example 3

What is the 23rd multiple of 2?

Answer: 46

The 23rd multiple of 2 means the sum of 23 two's. Rather than add 23 two's, we may use the fact that the 23rd multiple of 2 is the same number as the 2nd multiple of 23.

But the 2nd multiple of 23 is easy to find. It is the sum of two 23's, or:

$$\begin{array}{r} 23 \\ +\ \underline{23} \\ 46 \end{array}$$

In terms of tally marks, t! 2nd multiple of 23 is:

|||||||||||||||||||||||

|||||||||||||||||||||||

and this can be viewed as 23 columns each with 2 tall marks; that is, the 23rd multiple of 2.

Because of the importance of using multiples of whole numbers, we use special vocabulary and symbols to describe it. First of all, almost as the name seems to imply, the process of finding any multiple of any whole number is called *multiplication*.

Vocabulary for Multiplication

(1) *To indicate that we want the 4th multiple of 9, we write:*

$$9 \times 4$$

(2) *"X" is called a "times sign" and we read 9 X 4 as "nine times four".*

(3) *Since the 4th multiple of 9 is 36, we write:*

$$9 \times 4 = 36$$

(4) *When we write 9 X 4 we say that we're multiplying 9 by 4 or, equivalently, we're multiplying 4 by 9.*

(5) *When we write 9 X 4 = 36, we call 36 the* product *of 9 and 4. 9 and 4 are called the* factors.

We could use any whole numbers, but for the sake of illustration, we're using Example 2.

Notice that we write the multiple after we write the number. That is 4 X 9 woul mean the 9th multiple of 4.

However, since the 4th multiple of 9 is the same numbe as the 9th multiple of 4, t order isn't that critical.

In the "old days" we called 9 the multiplier *and 4 the* multiplicand. *That is:*

multiplier X multiplicand = product

It is convenient to know how to multiply one single-digit number by another. To this end, we can construct a multiplication table in a manner that is analogous to how we constructed the addition table. In fact, we use repeated addition to do multiplication.

Example 4

How much is 8 X 6?

8 X 6 means the 6th multiple of 8.

The 6th multiple of 8 means the sum of six 8's, or:

$$8 + 8 + 8 + 8 + 8 + 8$$

Using vertical form, we have:

$$\begin{array}{rrl} & 8 & \textit{(1st multiple of 8 or 8 X 1)} \\ + & 8 & \\ \hline & 16 & \textit{(2nd multiple of 8 or 8 X 2)} \\ + & 8 & \\ \hline & 24 & \textit{(3rd multiple of 8 or 8 X 3)} \\ + & 8 & \\ \hline & 32 & \textit{(4th multiple of 8 or 8 X 4)} \\ + & 8 & \\ \hline & 40 & \textit{(5th multiple of 8 or 8 X 5)} \\ + & 8 & \\ \hline & 48 & \textit{(6th multiple of 8 or 8 X 6)} \end{array}$$

Answer: 48

If you interpreted the Example as the 8th multiple of 6, you'd have obtained:

$$\begin{array}{rrl} & 6 & (1) \\ + & 6 & (2) \\ \hline & 12 & \\ + & 6 & (3) \\ \hline & 18 & \\ + & 6 & (4) \\ \hline & 24 & \\ + & 6 & (5) \\ \hline & 30 & \\ + & 6 & (6) \\ \hline & 36 & \\ + & 6 & (7) \\ \hline & 42 & \\ + & 6 & (8) \\ \hline & 48 & \end{array}$$

The key point is that 6 X 8 = 8 X 6 even though the addition forms look different.

Looking at Example 4 we see that it isn't as easy to write down the multiples of 8 as it was to write down, say, the multiples of ten. However, there are two single-digits that have convenient multiples. The two numbers are 0 and 1.

For example, to find the 6th multiple of 10, we'd simply write 6 and annex a zero.

Example 5

How much is 0 X 3?

Answer: 0

0 X 3 means the 3rd multiple of 0, or the sum of three 0's. But from addition we know that:

This is a "toughie" for tally marks. If we have 3 rows, each with no tally marks there are no tally marks to count.

$$\begin{array}{r} 0 \\ +\ 0 \\ \hline 0 \\ +\ 0 \\ \hline 0 \end{array}$$

No matter how many 0's we add, the sum is still 0. So any multiple of 0 is still 0.

Example 6

How much is 56 X 1?

Answer: 56

56 X 1 means the 1st multiple of 56.

But the 1st multiple of 56 is 56.

That is, when we count by 56's, the first number we come to is 56.

> *Note:*
>
> *56 X 1 = 1 X 56. 1 X 56 is the 56th multiple of 1, but when we count by 1's we get the whole numbers in sequential order.*

1 = 1
2 = 1 + 1
3 = 1 + 1 + 1
4 = 1 + 1 + 1 + 1
and so on.

Using 56 in Example 6 wasn't important. What is important is that every number is the 1st multiple of itself. By imitating the results of Examples 1 through 6 we can construct the multiplication table for single-digits. Leaving any additional details to you to verify, we have:

More abstractly, if n denotes any whole number: then n X 1 = n

X	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

For example to find 5 X 8 (the 8th multiple of 5):

(1) Find the row headed by "5"

(2) Move along this row until you're in the column headed by "8"

(3) Read the number in that location (40)

The above table, once we've noticed a few more properties of multiplication, will enable us to find the product of any two (or more) whole numbers.

Let's review the properties we already have:

<u>The Commutative Property For Multiplication</u>

For any two whole numbers:

(first number) X (second number) = (second number) X (first number)

In symbols, if m and n stand for any two whole numbers, $m \times n = n \times m$

<u>The Multiplicative Identity</u>

For any whole number n,

$n \times 1 = n$

These two properties are similar to ones we had in addition. Namely:
$m + n = n + m$
and
$n + 0 = n$

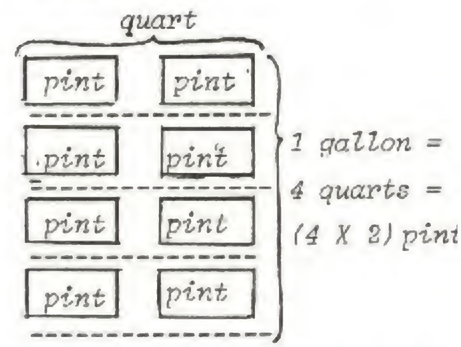
The key that enabled us to add more than two whole numbers was the <u>Associative Property</u>. As you might expect, there is also an associative

property for multiplication. The following example tries to illustrate this idea.

Example 7

There are 2 pints in a quart and 4 quarts in a gallon. How many pints are there in 3 gallons?

Answer: 24



Method 1

Since there are 2 pints in each quart, 4 quarts will contain 2 pints, 4 times. That is, there are 2 X 4 or 8 pints in 4 quarts. Since a gallon is 4 quarts, there are 8 pints in each quart.

Hence in 3 gallons there will be 8 pints, 3 times or 8 X 3 pints. Since 8 X 3 = 24, there are 24 pints in 3 gallons.

2 X 4 = 8 and 8 X 3 = 24 are found directly from our multiplication table.

In summary, we computed:

(2 X 4) X 3

Method 2

There are 4 quarts in a gallon, so in 3 gallons there are 4 quarts,3 times or(4 X 3) quarts. Since 4 X 3 = 12, there are 12 quarts in 3 gallons. Then since each quart has 2 pints, there are 12 X 2 or 24 pints in 3 gallons. *Since 12 X 2 = 2 X 12, we computed 2 X (4 X 3)*

That is, the 2nd multiple of 12 is 12 + 12 or 24. But our tables don't includ 12 X 2. Our table does include 4 X 3 = 12

If we compare Methods 1 and 2, we see that:

(2 X 4) X 3 = 2 X (4 X 3)

This result can be generalized as follows:

The Associative Property For Multiplication

If f, s, and t represent whole numbers, then:

$$(f \times s) \times t = f \times (s \times t)$$

This compares with the corresponding property of addition. Namely:
$(f + s) + t = f + (s + t)$

As we shall soon see, the associative property for multiplication will be quite helpful to us. But there are still other properties that we'd like to develop first. For one thing, let's see how we can treat other than single-digit multiples of a whole number.

Example 8

What is the 10th multiple of 8?

Answer: 80

In terms of multiplication we want 8 X 10. But the 10th multiple of 8 is the same number as the 8th multiple of 10; and in Module 2 we saw that to get the 8th multiple of 10, we simply wrote 8 and annexed one 0 to get 80.

Notice that the table stops at 9 X 9 (although up to recent times, the multiplication table went through 12 rather than through 9. We'll see why in the next module.)

The idea in Example 8 applies to any power of ten and any whole number. For example:

Example 9

What is the 100th multiple of 37?

Answer: 3,700

The 100th multiple of 37 is the same number as the 37th multiple of 100. Using the idea developed in Module 2 we find this

by writing 37 and annexing two 0's to get 3700. Locating the comma in the proper place, we write the answer as 3,700.

In the language of multiplication, Example 9 says that 37 X 100 = 3,700. *More generally,*

37×10^n is 37 followed by n zeros.

Recall from Module 1 that 10^n is a 1 followed by n zeros.

The associative property for multiplication allows us to extend our above result to multiples of powers of ten.

Example 10

How much is 6 X 2,000?

Answer: 12,000

2,000 is the 2nd multiple of 1,000 or 1,000 X 2. Hence we may write:

6 X 2,000 =

6 X (1,000 X 2) =

6 X (2 X 1,000) = *We used the commutative pro[*

(6 X 2) X 1,000 = *We used the associative pro[*

12 X 1,000 = *6 X 2 = 12 comes from the multiplication table.*

12,000 *To multiply 12 by 1,000 we write 12 and annex 3 zeros.*

Note:

If we write Example 2 as 6 X 2 thousand, the associative property allows us to read it as (6 X 2) thousand or 12,000.

The key point is that excep[for the 0's all the multipl[cation is from the table for single digits.

Example 10 shows us that using the table alone, we can find the 2,000th multiple of 6. We could just as easily found the 2,000th multiple of 600.

Example 11

Use the result of Example 10 to find the product of 600 and 2,000. That is, how much is 600 X 2,000?

Answer: 1,200,000

In the last example we saw that

6 X 2,000 = 12,000. Hence by the commutative property for multiplication 2,000 X 6 = 12,000.

Now we can write 600 X 2,000 as:

600 X 2,000 =	
2,000 X 600 =	*by the commutative property*
2,000 X (100 X 6) =	*600 is the 6th multiple of 100*
2,000 X (6 X 100) =	*by the commutative property*
(2,000 X 6) X 100 =	*by the associative property*
12,000 X 100 =	*by Example 10*
1200000 =	*To multiply 12,000 by 100 we write 12,000 and annex 2 zeros--then relocate the commas*
1,200,000	

Note The Short-Cut

To multiply 600 by 2,000 we first multiplied 6 by 2 to get 12.

And to do this we need only the multiplication table.

Then we counted the ending-zeros in each factor and added them:

$$\underset{1\ 2}{6\,00} \qquad \underset{3\ 4\ 5}{2,000}$$

We then annex this number of 0's to 12 to get $\underset{1\,2\,3\,4\,5}{1200000}$

Then we relocate the commas in their proper place to get 1,200,000

This is an interesting departure from addition. For example we couldn't use 6 + 2 = 8 to add 600 and 2,000. But we can use 6 X 2 = 12 to multiply 600 by 2,000. *In fact: 6 hundred X 2 thousand = 12 hundred-thousand. We multiply the adjectives (6 and 2) and we also multiply the nouns (hundred and thousand)*

The short cut makes it rather easy to multiply any multiple of a power of ten by any other multiple of a power of ten.

Example 12

How much is 7,000,000 X 80,000?

Answer: 560,000,000,000 (560 billion)

Let's use the shortcut. We first multiply 7 by 8 to get 56. The first factor ends in 6 zeros and the second factor ends in 4 zeros. So the total number of ending 0's is 10. Therefore we annex 10 zeros to 56 to get:

7 X 8 = 56 is in the table

7,000,000	80,000
123 456	1 234

$$5\ 6\ \underset{1}{0}\ \underset{2}{0}\ \underset{3}{0}\ \underset{4}{0}\ \underset{5}{0}\ \underset{6}{0}\ \underset{7}{0}\ \underset{8}{0}\ \underset{9}{0}\ \underset{10}{0}$$

and we place the commas to get:

5 6 0, 0 0 0, 0 0 0, 0 0 0

This is an amazingly quick way to find the 80,000th multiple of 7,000,000!

The only remaining serious problem is how to handle factors that aren't multiples of powers of ten. To handle this problem we need a property that applies when addition and multiplication appear in the same problem.

For example rather than 600 X 200 we might have 613 X 257. Is there a convenient way to find the 257th multiple of 613?

Example 13

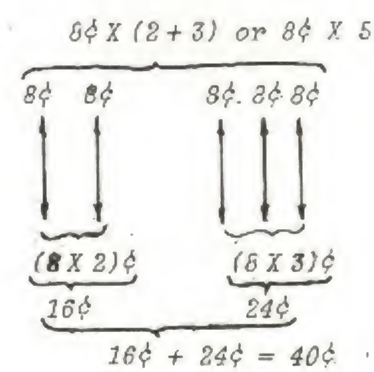
You buy 2 pencils for 8¢ each. Later you return and buy 3 more pencils, again at 8¢ each. How much did you pay for the pencils altogether?

Answer: 40¢

Method 1:

For the first 2 pencils you paid 8¢ twice. That is: 8¢ X 2 = 16¢. Then you paid 8¢ three times for the other 3 pencils. That cost you 8 X 3¢ or 24¢ more. Altogether you paid 16¢ + 24¢ or 40¢. *In symbols the number of cents you paid was:*

(8 X 2) + (8 X 3)



Method 2

Altogether we bought 2 + 3 or 5 pencils. Each pencil cost 8¢. So we spent 8¢, 5 times. Since 8 X 5 = 40, we paid 40¢ for the 5 pencils. *In symbols, the number of cents you paid was 8 X (2 + 3).*

If we compare the answers we found by Methods 1 and 2 we see that:

$$8 \times (2 + 3) = (8 \times 2) + (8 \times 3)$$

That is, to multiply 8 by the sum of 2 and 3; we first multiply 8 by 2, then we multiply 8 by 3, and then we add these two products. This result can be generalized as follows:

What makes this different from anything we've done so far in Modules 2 and 3 is that we now have addition and multiplication in the same problem.

The Distributive Property

For any three whole numbers:

first X (second + third) =

(first X second) + (first X third)

More formally, we sometimes call this the distributive property of multiplication over addition to indicate that the multiplier is distributed with each summand in the multiplicand.

Example 14

Use the distributive property to compute the value of 2 X (10 + 4).

Answer: 28

$$2 \times (10 + 4) =$$
$$\underbrace{(2 \times 10)}_{20} + \underbrace{(2 \times 4)}_{8} =$$
$$20 + 8 =$$
$$28$$

If we had not been told to use the distributive property, we'd have probably replace 10 + 4 by 14 to obtain:
2 X (10 + 4) =
2 X 14 =
14 X 2 =
14 + 14 =
28
But we simply wanted an "easy" problem to use in order to check the distributive property.

Note that this result is easy to visualize in terms of tally marks. Namely, write two rows each with 10 + 4 (14) tally marks:

$$\underbrace{\begin{matrix} |\,|\,|\,|\,|\,|\,|\,|\,|\,| \\ |\,|\,|\,|\,|\,|\,|\,|\,|\,| \end{matrix}}_{\textit{2 tens}} \quad (+) \quad \underbrace{\begin{matrix} |\,|\,|\,| \\ |\,|\,|\,| \end{matrix}}_{\textit{2 fours}}$$

It might be helpful to rewrite Example 14 in a way that emphasizes how we used the multiplication table for single digits. Namely:

$2 \times (10 + 4) =$

$2 \times (1\ \textit{ten} + 4\ \textit{ones}) =$

$(2 \times 1\ \textit{ten}) + (2 \times 4\ \textit{ones}) =$

$(2 \times 1)\ \textit{tens} + (2 \times 4)\ \textit{ones} =$

$2\ \textit{tens} + 8\ \textit{ones} =$

$20 + 8 =$

28

This emphasizes that in effect we only had to know that 2 X 1 = 2 and 2 X 4 = 8

We can abbreviate the above procedure if we elect to use the same vertical form that we used for addition. For example, we could begin by writing

$$\begin{array}{r} 1\ 4 \\ \times \quad 2 \\ \hline \end{array}$$

This represents $\begin{array}{r} 14 \\ +\ 14 \\ \hline \end{array}$

Then:

(1) Multiply 2 by 4 and write the product under the 2:

$$\begin{array}{r} 1\ 4 \\ \times \quad 2 \\ \hline 8 \end{array}$$

This takes the place of saying "4 + 4 = 8"

That is, 4 ones twice is the same as 8 ones.

(2) Now multiply 2 by 1 and write the answer under the 1:

$$\begin{array}{r} 1\ 4 \\ \times \quad 2 \\ \hline 2\ 8 \end{array}$$

We place the product under the 1 to hold the proper place. That is, we really have 2 X 1 <u>ten</u> *= 2* <u>tens</u>.

Note that the 2 multiplies <u>each</u> digit--the 4 and the 1--of 14.

This is different from addition where we only add 2 to 4. Namely:

$$\begin{array}{r} 1\ 4 \\ +\quad 2 \\ \hline 1\ 6 \end{array}$$

In other words, 14 + 2 can be written as (10 + 4) + 2 or 10 + (4 + 2). On the other hand 14 X 2 becomes (10 + 4) X 2 and this requires the distributive property rather than the associative property because addition and multiplication appear together.

Although we stated the distributive property in terms of only two summands in the multiplicand, we may apply it to any number of summands. This is illustrated in the next example.

Example 15

Find the 2nd multiple of 1,234.

Answer: 2,468

Method 1

We want the sum of two 1,234's. using the vertical form for addition we obtain:

$$\begin{array}{r} 1,\ 2\ 3\ 4 \\ +\ \underline{1,\ 2\ 3\ 4} \\ 2,\ 4\ 6\ 8 \end{array}$$

We do this example by addition so that we can compare it with multiplication. If we had a greater multiple of 1,234 Method 1 could become very cumbersome.

Method 2

In the language of multiplication the problem is 1,234 X 2, or by the commutative property, 2 X 1,234. This can be written as 2 X (1,000 + 200 + 30 + 4) or:

2 X (1 *thousand* + 2 *hundred* + 3 *tens* + 4 *ones*) =

Now we "distribute" the 2.

(2 X 1) *thousand* + (2 X 2) *hundred* + (2 X 3) *tens* + (2 X 4) *ones* =

2,000 + 400 + 60 + 8 =

2,468

Method 3

We redo Method 2 using the vertical form:

Step 1: *Write*

$$\begin{array}{r} 1\ 2\ 3\ 4 \\ X\ \underline{\ 2} \end{array}$$

Step 2: *Multiply 2 by 4 and write the product under the 4.*

$$\begin{array}{r} 1\ 2\ 3\ 4 \\ X\ \underline{\ 2} \\ 8 \end{array}$$

That is, 2 X 4ones = 8 ones

Step 3: *Next multiply (the bottom) 2 by 3 and write the product under the 3.*

$$\begin{array}{r} 1\ 2\ 3\ 4 \\ X\ \underline{\ 2} \\ 6\ 8 \end{array}$$

We're multiplying 2 by 3, but the 3 stands for 3 tens. 2 X 3 tens = 6 tens. Placing the 6 under the 3 puts the 6 in the tens-place.

Step 4: *Next multiply 2 by 2 and write the product under the (top) 2.*

$$\begin{array}{r} 1\ 2\ 3\ 4 \\ X\ \underline{\ 2} \\ 4\ 6\ 8 \end{array}$$

That is, we really have 2 X 2 hundred = 4 hundred. So we've placed the 4 in the hundreds-place.

Step 5: *Finally multiply (the bottom) 2 by 1 and write the answer under the 1.*

$$\begin{array}{r} 1\ 2\ 3\ 4 \\ \times \quad\quad\ \ 2 \\ \hline 2\ 4\ 6\ 8 \end{array}$$

The 1 is in the thousands-place so by writing the 2 under the 1 it means 2 thousand; which checks with the fact that 2 X 1 thousand is 2 thousand.

The multiplication is now complete because the (bottom) 2 has now multiplied each digit in 1,234, Had we actually used Method 3 to do this example, all we'd have written is Step 5.

Sometimes we have to carry in the vertical form of multiplication just as we did in the vertical form of addition.

Example 16

Find the 3rd multiple of 24?

Answer: 72

Method 1:

Again for illustration purposes, let's use the vertical form of addition. The 3rd multiple of 24 means the sum of three 24's, or:

$$\begin{array}{r} 1 \\ 2\ 4 \\ +\ 2\ 4 \\ +\ 2\ 4 \\ \hline 7\ 2 \end{array}$$

Method 2:

We could use the commutative and distributive properties to obtain:

24 X 3 =

3 X 24 =

3 X (20 + 4) =

$\underbrace{(3 \text{ X } 20)}_{60} + \underbrace{(3 \text{ X } 4)}_{12} =$

72

We can also, by the commuta tive property, write:
(20 + 4) X 3 =

(20 X 3) + (4 X 3)

So, in effect, we multiplie 3 by 2 and 3 by 4.

Method 3:

Use the vertical form in Method 2.

Step 1: *Write*

$$\begin{array}{rcc} & 2 & 4 \\ \times & & 3 \\ \hline \end{array}$$

Step 2: *Multiply 3 by 4 to get 12. But we can have only one digit per place value. Therefore write the 2 (ones) under the 4 and add one more ten by "carrying" the 1 to the tens-place:*

$$\begin{array}{rcc} & 1 & \\ & 2 & 4 \\ \times & & 3 \\ \hline & & 2 \end{array}$$

If the place values were written, we could have:

$$\begin{array}{rcc} & \underline{\text{tens}} & \underline{\text{ones}} \\ & 2 & 4 \\ \times & & 3 \\ \hline & 6 & 12 \\ \hline & 7 & 2 \\ \hline \end{array} \quad \begin{array}{l} (12 = 1 \text{ ten} \\ \qquad + \\ \quad 2 \text{ ones}) \end{array}$$

Step 3: *Multiply the 3 by 2 and add the 1 we carried to get 7. Write the 7 under the (top) 2.*

$$\begin{array}{rcc} & 1 & \\ & 2 & 4 \\ \times & & 3 \\ \hline & 7 & 2 \end{array}$$

The problem is now completed because the 3 has multiplied each digit in the 24.

That is, 2 tens X 3 = 6 tens and 6 tens + 1 ten = 7 tens By placing the 7 under the top 2 we've placed the 7 in the tens-place.

Method 3 is really a more compact version of Method 1. Namely in adding three 24's we add three 20's and three 4's.

If in a multiplication problem, both factors have more than a single-digit; we can use a more general form of the distributive property. Let's illustrate this idea by finding the product of 46 and 37. We have:

46 X 37 =

(40 + 6) X (30 + 7) [1]

Treat (40 + 6) as one number and distribute it with the 30 and 7 to get:

[(40 + 6) X 30] + [(40 + 6) X 7]

That is, we know that 46 X (30 + 7) = (46 X 30) + (46 X 7) All we're doing is writing 46 as 40 + 6

Next apply the distributive property to each expression in brackets. Namely:

[(40 X 30) + (6 X 30)] + [(40 X 7) + (6 X 7)]

Since addition has the associative property, we may omit the brackets and write:

(40 X 30) + (6 X 30) + (40 X 7) + (6 X 7)   [2]

If we compare [1] and [2], we see that we multiplied each summand in (40 + 6) by each summand in (30 + 7). *More generally, each summand in the multiplier must always multiply each summand in the multiplicand.*

Vertical form makes the above procedure much easier to keep track of. Let's see how it works step-by-step.

Step 1: *Write the two factors, one underneath the other so that the ones-places line up.*

$$\begin{array}{rcc} & 4 & 6 \\ \times & \underline{3} & \underline{7} \end{array}$$

Step 2: *Proceed as in the last few examples and multiply 6 by 7 to get 42. Write the 2 under the 7 and carry the 4.*

We're first going to multiply each digit in 46 by 7. (This is our 6 X 7 term)

$$\begin{array}{rcc} & 4 & \\ & 4 & 6 \\ \times & \underline{3} & \underline{7} \\ & & 2 \end{array}$$

Step 3: *Now multiply 7 by 4 to get 28 and add the 4 you carried to get 32.*

Actually, it's 28 tens + 4 tens. Notice that since the 4 is in the tens-place we've computed the 40 X 7 term.

$$\begin{array}{ccc} & 4 & \\ & 4 & 6 \\ \times & \underline{3} & \underline{7} \\ 3 & 2 & 2 \end{array}$$

Step 4: *Multiply 3 by 6 to get 18. Since the 3 stands for 3 tens, the product is 18 tens or 180. Hence we want the 8 to be in the tens-place. We accomplish this by writing the 8 under the 3 in 37, and we carry the 1 in 18.*

Now we're going to multiply each digit in 46 by 3.

$$\begin{array}{cccc} & 1 & & \\ & 4 & 6 & \\ \underline{\times} & \underline{3} & \underline{7} & \\ 3 & 2 & 2 & \\ & 8 & (0) & \end{array}$$

We can omit the 0 because the 7 in 37 is already holding the ones-place.

Since the 3 stands for 3 tens, this is our 30 X 6 or 6 X 30 term.

Step 5: *We now multiply 3 (tens) by 4 (tens) and add the 1 (hundred) we carried to get (3 X 4) + 1 or 12 + 1 or 13.*

```
        1
        4  6
   X    3  7
     3  2  2
  1  3  8 (0)
```

Remember that 3 tens X 4 tens is 12 hundreds. Placing the 3 under the 3 in 322 means we've put it in the hundreds-place. That is, the 3 in 322 holds the hundreds-place.

When we multiplied 4 by 3 it was really 4 tens by 3 tens, and this accounts for the 40 X 30 term.

Step 6: *Now perform addition on the two numbers between the lines--322 and 1,380.*

```
           4  6
     X     3  7
        3  2  2    (7 X 46)
 +   1  3  8 (0)   (30 X 46)
     1  7  0  2    (37 X 46)
```

So what we've actually done is compute the 37th multiple of 46.

From now on, let's use only the vertical form.

Example 17

Find the 21st multiple of 345.

Answer: 7,245

The 21st multiple of 345 means 345 X 21.

Using the vertical form we have:

Remember the place values!

	hundreds	tens	ones
	3	*4*	*5*
X		*2*	*1*

Step 1: *Write:*

```
     3  4  5
 X      2  1
```

Step 2: *Multiply each digit in 345 by 1.*

```
     3  4  5
 X      2  1
     3  4  5
```

Step 3: *Multiply 2 by 5 to get 10. Write the 0 under the 4 and carry the 1.*

This is really 2 tens X 5 or 10 hundreds.

```
        1
     3  4  5
 X      2  1
     3  4  5
        0 (0)
```

In Steps 3, 4, and 5 we're multiplying each digit in 345 by 2.

Step 4: *Now multiply 2 by 4 to get 8 and add the 1 you carrried to get 9.*

It's really 2 tens X 4 tens which is 8 hundreds. The 3 in 345 is in the hundreds-place. So placing the 9 under the 3 puts it in the hundreds-place

```
        1
     3  4  5
 X      2  1
     3  4  5
     9  0 (0)
```

Step 5: *Multiply the 2 by 3 to get 6, and place the 6 to the left of the 9.*

$$\begin{array}{ccccc}
 & & & 1 & \\
 & & 3 & 4 & 5 \\
X & & & 2 & 1 \\ \cline{3-5}
 & & 3 & 4 & 5 \\
 & 6 & 9 & 0 & (0) \\ \cline{2-5}
\end{array}$$

We're really multiplying 2 tens by 3 hundreds and this is 6 thousand. Since the 9 is in the hundreds-place, we've put the 6 in the thousands-place.

Step 6: *Perform the addition:*

$$\begin{array}{ccccc}
 & & 3 & 4 & 5 \\
X & & & 2 & 1 \\ \cline{3-5}
 & & 3 & 4 & 5 \\
+ & 6 & 9 & 0 & (0) \\ \cline{2-5}
 & 7, & 2 & 4 & 5
\end{array}$$

In terms of addition:

$$\left.\begin{array}{l}\text{ten } 345\text{'s} = 3,450 \\ \text{ten } 345\text{'s} = 3,450\end{array}\right\} 6,900$$
$$\begin{array}{l}\text{one } 345 \quad = \quad 345 \\ \hline \qquad\qquad = 7,245\end{array}$$

Note that when we actually do this problem ourselves, only Step 6 appears. We also tend to omit the 0 in parentheses. So we usually write it as:

$$\begin{array}{cccc}
 & 3 & 4 & 5 \\
X & & 2 & 1 \\ \cline{2-4}
 & 3 & 4 & 5 \\
6 & 9 & 0 & \\ \hline
7, & 2 & 4 & 5
\end{array}$$

(The 5 holds the ones-place. So we don't need 0 as a place holder.)

Remember what this procedur really means. We've found a quick way to find the sum of twenty one 345's, making use of the single-digit multiplication table.

Additional exercises are left for the Self-Test. But before we conclude this module we'd like to talk about a companion topic called *factoring*. Up to now we've started with the factors and found the product. *Factoring is when we start with the product and want to find the factors.*

Example 18

Is 5 a factor of 24?

Answer: No

What we're really asking is whether there is a whole number that can replace the blank in:

5 X ___ = 24

Notice how "factor" and "multiple" complement each other. That is, 5 is a fac tor of 24 means the same thing as 24 is a multiple of 5.

If we count by 5's we get:

5, 10, 15, 20, 25, . . . This tells us that 24 is between the 4th multiple of 5 (20) and the 5th multiple of 5 (25). In particular, 24 is not a multiple of 5. Equivalently, 5 is not a factor of 24.

In terms of tally marks, if we start with 24 tally marks and arragne them in rows of five, we get more than 4 complete rows but less than 5:

```
| | | | | (5)
| | | | | (10)
| | | | | (15)
| | | | | (20)
| | | |   (24)
```

Example 19

Is 6 a factor of 24?

Answer: Yes

For 6 to be a factor of 24, 24 must be a multiple of 6 . So we count by 6's and see if 24 is one of the numbers we get:

6, 12, 18, <u>24</u>

We've found that 24 is the 4th multiple of 6. Hence 6 is a factor of 24.

Again in terms of tally marks we start with 24 and arrange them in rows of 6 each:

```
| | | | | | (6)
| | | | | | (12)
| | | | | | (18)
| | | | | | (24)
```

and we get exactly 4 complete rows.

Example 19 tells us that *one* way to factor 24 is as 6 X 4. But as indicated below there are other ways to factor 24. Namely:

24 = 24 X 1

That is, 24 is the 1st multiple of 24 and the 24th multiple of 1.

24 = 12 X 2

That is, 24 is the 12th multiple of 2 and the 2nd multiple of 12.

Any number that ends in 0, 2, 4, 6, or 8 has 2 as a factor. We'll discuss this in more detail in the next module.

24 = 8 X 3

That is, 24 is the 8th multiple of 3 and the 3rd multiple of 8.

24 = 6 X 4

That is, 24 is the 6th multiple of 4 and the 4th multiple of 6.

These results can be verified using the multiplication table. We can also, for example, count by 8's to get 8, 16, <u>24</u>

So while it is well-defined to ask for the product of 6 and 4, it is ambiguous to ask for *<u>the two</u>*

whole numbers whose product is 24. What we often do is what we did in Examples 18 and 19. We pick a particular number and see if it is a factor of the given number.

Example 20

Is 13 a factor of 57?

Answer: No

Let's count by 13's. One of two things must happen. Either 57 will be one of these numbers (in which case 13 is a factor of 57) or we'll get to a number that's greater than 57 without 7 appearing (in which case, 13 is not a factor of 57. So let's count by 13's.

That is, we'll "jump" from a number less than 57 to a number greater than 57.

13, 26, 39, 52, 65

We're adding 13 each time. In multiplication form: 13 X 1 = 13, 13 X 2 = 26, 13 X 3 = 39, 13 X 4 = 52, and 13 X 5 = 65. Since 57 is between 52 and 65, it is not a multiple of 13. In fact 57 is between the 4th and 5th multiples of 13.

Note that counting by 13's tells us many numbers that do not have 13 as a factor. Of all the whole numbers between 1 and 65 only 13, 26, 39, 52, and 65 are multiples of 13. Hence 13 cannot be a factor of any other numbers between 1 and 65.

Special Note:

For any whole number n, n X 0 = 0. Hence n is a factor of 0. In other words, 0 is a multiple of every whole number. For this reason we usually start with 1 when we study multiples.

For example, 0 is the 0th multiple of 1, 2, 3, 4, 5, and so on.

The method used in Example 20 is fine for relatively small multiples. But suppose we had wanted to find whether 13 was a factor of 2,821. At best, it would be tedious to list the multiples of 13 until we either came to 2,821 or exceeded it. Fortunately there is a "neater" way.

We proceed exactly as we did in Example 20, but we stop after we've listed the 9th multiple of 13.

13 X 1 = 13
13 X 2 = 26
13 X 3 = 39
13 X 4 = 52
13 X 5 = 65
13 X 6 = 78
13 X 7 = 91
13 X 8 = 104
13 X 9 = 117

We could simply write: 13, 26, 39, 52, 65, 78, 91, 104, 117; but we want to emphasize the multiplication language.

Just as we explained in our construction of the multiplication table, we never have to compute more than single-digit multiples of any number.

For example, once we know that 13 X 4 = 52, we also know that 13 X 40 = 520, 13 X 400 = 5,200 and so on.

Looking at the first two digits of 2,821 and comparing this with our "13-table", we notice that 2,821 is between 2,600 and 3,900. This tells us that 2,821 is greater than the 200th multiple of 13 but less than the 300th multiple of 13.

See? Since 13 X 2 = 26, 13 X 200 = 2,600--and this is less than 2,821. Similarly, 13 X 300 = 3,900--and this is greater than 2,821.

To see how many more 13's we need, we *subtract* the 200th multiple of 13 (2,600) from 2,821. Using the vertical form, we get:

$$\begin{array}{rl} & 2{,}821 \\ - & \underline{2{,}600} \ \ \textit{(13 X 200)} \\ & 221 \end{array}$$

The 10th multiple of 13 is 130 and the 20th multiple of 13 is 260. Hence 221 is between 130 and 260. This leads to:

$$\begin{array}{rl} & 2{,}821 \\ - & \underline{2{,}600} \ \ \textit{(13 X 200)} \\ & 221 \\ - & \underline{130} \ \ \textit{(13 X 10)} \\ & 91 \end{array}$$

Now our "13-table" shows us that 91 = 13 X 7. So:

$$\left.\begin{array}{rl} & 2{,}821 \\ - & \underline{2{,}600} \ \ \textit{(13 X 200)} \\ & 221 \\ - & \underline{130} \ \ \textit{(13 X 10)} \\ & 91 \\ - & \underline{91} \ \ \textit{(13 X 7)} \\ & 0 \end{array}\right\}$$

So 2,821 is the 217th multiple of 13. That is: 2,821 = 13 X 217

We could have proceeded more gradually by using only powers of ten. That is:
13 X 10 = 130
13 X 100 = 1,300
13 X 1,000 = 13,000
Since 2,821 is between 1,300 and 13,000, we know that 2,821 is more than the 100th multiple of 13 but less than the 1,000th multiple of 13;

$$\begin{array}{rl} & 2{,}821 \\ - & \underline{1{,}300} \ \ \textit{(100 thirteens)} \\ & 1{,}521 \\ - & \underline{1{,}300} \ \ \textit{(100 thirteens)} \\ & 221 \\ - & \underline{130} \ \ \textit{(10 thirteens)} \\ & 91 \\ - & \underline{91} \ \ \textit{(7 thirteens)} \\ & 0 \end{array}$$

See how we use the distributive property?
(13 X 200) + (13 X 10) + (13 X 7) = 13 X (200 + 10 + 7) = 13 X 217

In less formal terms, we showed that to get 2,821 we had to add 200 thirteens, 10 more thirteens, and then 7 more thirteens; giving us a total of 217 thirteens.

200 thirteens
10 thirteens
7 thirteens
217 thirteens

The process of finding whether one number (in the last example, 13) is a factor of another (2,821) is called *division*. Like the other operations of arithmetic, division brings with it, its own symbolism and vocabulary.

****

Special Vocabulary for Division

To indicate that we want to solve the problem:

$$13 \times \_\_\_\_ = 2,821$$

we write

$$2,821 \div 13 = \_\_\_\_$$

which we reads as "2,821 divided by 13 is equal to what number?"

To utilize what we did in our earlier discussion, we often represent 2,821 ÷ 13 in the form:

$$13\overline{)2,821}$$

which we read as "13 'goes into' 2,821"

In this case we call 13 the divisor *and 2,821 the* dividend. *The answer is called the* quotient.

****

We're using 13 and 2,821 only for illustrative purposes.

Compare this with addition and subtraction. If we had $4 + \_\_ = 12$ we wrote $12 - 4 = \_\_$. Here we say that $4 \times \_\_ = 12$ can be written as $12 \div 4 = \_\_\_\_$

That is:

$$\text{divisor}\overline{)\text{dividend}}^{\;\text{quotient}}$$

or

divisor X quotient = dividend

In terms of the "goes into" language there is a very nice compact way of summarizing what we did in order to find out whether 13 was a factor of 2,821.

Namely:

Step 1:

To see if 13 is a factor of 2,821 first write:

```
13)2 8 2 1
```

The comma in 2,821 is omitted to make it easier to interpret the next few steps.

Step 2:

We read the digits in 2 8 2 1 until we get to a number that's greater than 13 (the divisor). That is 2 is less than 13 but 28 is greater than 13.

```
13)2 8 2 1
     ↑
```

So pretend the problem is

```
13)28
```

Step 3:

Find the number of times 13 "goes into" 28. That is, use the "13-table" to see that 2 is the greatest multiple of 13 that's less than 28. Write the 2 in the quotient <u>above the 8</u>. *(The arrow tells us where to place the 2.*

That is, 13 X 2 = 26 which is less than 28 while 13 X 3 = 39 which is more than 28.

```
     2
13)2 8 2 1
```

The 8 holds the hundreds-place. So by placing the 2 over the 8, the 2 stands for 2 hundred.

Step 4:

Multiply 13 by 2 to get 26 (you can get this from your "13-table" right away). Subtract the 26 from 28.

The advantage of preparing the "13-table" is so that you don't have to stop and do the multiplication now.

```
     2
13)2 8 2 1
  -2 6
     2
```

It may not look it but we are really subtracting 26 hundred from 28 hundred.

Step 5:

"Bring down" the next digit (2) from the dividend.

```
     2
13)2 8 2 1
  -2 6 ↓
     2 2
```

Step 6:

Repeat the cycle starting with Step 2. Since 22 is already greater than 13 but less 26. So 13 goes into 22 1 time. So put a 1 above the 2nd 2 in the dividend.

Whenever we bring a number down, a number must eventually be placed above it in the quotient.

```
      2 1
13)2 8 2 1
 - 2 6
     2 2
```

Then multiply 13 by 1 to get 13 and subtract this from 22 to get:

```
      2 1
13)2 8 2 1
 - 2 6
     2 2
  -  1 3
       9
```

Step 7:

Bring down the next (and last) digit (1) from the dividend and repeat the cycle. That is, from our "13-table" we see that 13 goes into 91 7 times. We multiply 13 by 7 to get 91 and then subtract 91 from 91 to get 0.

```
      2 1 7
13)2 8 2 1
 - 2 6
     2 2
  -  1 3
       9 1
    -  9 1
         0
```

To emphasize how we were doing "rapid" subtraction, notice:

```
      2 1 7
13)2 8 2 1
 - 2 6 - -  (200 thirteens)
     2 2 1
  -  1 3 -  (10 thirteens)
       9 1
    -  9 1  (7 thirteens)
         0
```

(If at this step the difference had not been 0, we'd conclude that 13 wasn't a factor of 2,821)

Step 8:

We've now shown that 217 X 13 = 2,821. We check our answer by multiplying 217 by 13 to ensure that the product is 2,821.

```
    2 1 7
X     1 3
    6 5 1
+ 2 1 7
  2,8 2 1
```

Special Vocabulary

In the language of division once we know that 13 is a factor of 2,821 we say that 2,821 is divisible by 13.

Example 21

Is 13,320 divisible by 72?

Answer: Yes

This is the same as asking whether 72 is a factor of 13,320 . Let's try the "short form". We write:

```
     ______
72)1 3 3 2 0
       ↑
```

(1 and 13 are less than 72. 133 is greater than 72. Hence the placement of the arrow.)

```
       1
     ______
72)1 3 3 2 0
   - 7 2
   -----
     6 1
```

72 X 1 is less than 133. 72 X 2 is greater than 133.

```
       1 8
     ______
72)1 3 3 2 0
   - 7 2↓
   -------
     6 1 2
   - 5 7 6
     -----
       3 6
```

From the table, 612 is between 72 X 8 (576) and 72 X 9 (648)

```
       1 8 5
     ______
72)1 3 3 2 0
   - 7 2   ↓
   -----
     6 1 2
   - 5 7 6
     -------
       3 6 0
     - 3 6 0
       -----
           0
```

From the table, 72 X 5 = 360

(IF YOU UNDERSTAND THE PROCEDURE, THIS IS THE ONLY STEP YOU WRITE.)

The "72-Table":

72 X 1 = 72
72 X 2 = 144
72 X 3 = 216 (We're adding
72 X 4 = 288 by 72's)
72 X 5 = 360
72 X 6 = 432
72 X 7 = 504
72 X 8 = 576
72 X 9 = 648

We could get these results without listing the entire "72-table". But for illustrative purposes it is convenient to have the table.

Check:

```
        1 8 5
        X 7 2
        -----
        3 7 0
  + 1 2 9 5
    ---------
    1 3,3 2 0
```

So 13,320 is the 185th multiple of 72.

This means that 13,320 is divisible by 72.

By way of summary:

```
   13,320
 -  7,200  (100 seventy-two's)
   ------
    6,120
 -  5,760  (80 seventy-two's)
    -----
      360
    - 360  (5 seventy-two's)
      ---
        0
```

72 X 1 = 72 means
72 X 100 = 7,200

72 X 8 = 576 means
72 X 80 = 5,760

Example 22

Is 5,700 divisible by 72?

Answer: No

We proceed as in Example 21.

```
      7
72)5 7 0 0
 - 5 0 4
     6 6
```

Don't write

```
 7
72)5 7 0 0
```

because 57 is still less than 72. The 7 must be placed above the first 0.

```
      7 9
72)5 7 0 0
 - 5 0 4 ↓
     6 6 0
   - 6 4 8
       1 2
```

We're using the same "72-table" that we used in Example 21.

Note:

The process ends when there are no more digits in the dividend to "bring down". If the final subtraction gives us any number other than 0, the dividend isn't divisible by the divisor. The answer to the final subtraction is called the remainder. We might write:

What we've shown in this example is that 5,700 is between the 79th and 80th multiples of 72.

```
      7 9 R = 12
72)5 7 0 0
 - 5 0 4
     6 6 0
   - 6 4 8
       1 2
```

We usually say this as: "72 goes into 5,700; 79 times with a remainder of 12"

Check:

```
     7 9
   X 7 2
   1 5 8
 5 5 3
 5,6 8 8
   + 1 2
 5,7 0 0
```

When there's a nonzero remainder, we still check by multiplying the quotient by the divisor; but then we have to add the remainder.

An Interpretation

Suppose we had 5,700 books and we could put 72 books in every carton. To ship the books, we'd fill 79 boxes completely and have 12 books left to put in the 80th carton.

As you may sense from Example 22, if you pick a whole number at random, there's a good chance that it won't be divisible by 72. Such cases are very important in mathematics and we shall begin discussing them in the next module. For now we simply want to mention a few pitfalls and hints for division.

Example 23

By what must we multiply 72 to obtain 7,488 as the product?

Answer: 104

We want to divide 7,488 by 72. To this end we write:

```
72)7 4 8 8
     ↑
```

and we proceed as in the previous two examples.

```
      1
72)7 4 8 8
 - 7 2 ↓
     2 8
```

For every digit we bring down from the dividend, there must be a digit placed above it in the quotient. The fact that 28 is less than 72 means that we must place a 0 above the first 8 before we bring down the second 8!

```
     1 0 4
72)7 4 8 8
 - 7 2   ↓
     2 8 8
   - 2 8 8
         0
```

You could have said that 72 X 0 = 0 and written:

```
     1 0
72)7 4 8 8
 - 7 2   |
     2 8 |
   - 0   ↓
     2 8 8
```

Since the remainder is 0, we say that 7,488 is divisible by 72. In particular, we showed that 7,488 is the 104th multiple of 72.

Check:

```
      1 0 4
      X 7 2
      2 0 8
  + 7 2 8
    7,4 8 8
```

If by mistake you omitted the 0 and got 14 as the quotient, the check would not work. Namely,

```
    72
  X 14
   288
 + 72
 1,008
```

In fact we know that 72 X 100 is 7,200 and this tells us that the quotient must be greater than 100.

Now a final hint--Many people find it tedious to write out the table every time they have to do a division problem. It is relatively easy to guess the digit that must be placed in the quotient if we look at two guidelines. Namely:

(1) Suppose you guess too low.

```
       3
72)3 0 2 4
 - 2 1 6
     8 6
```

From the "72-table" we can see that 302 is between the 4th and 5th multiples of 72.

The clue is that after we subtracted <u>but before we brought down the next digit</u> *the answer was greater than the divisor. This means that we could have subtracted at least one more 72. Indeed 86 - 72 = 14 and we see:*

```
       4
72)3 0 2 4
 - 2 8 8
     1 4
```

In fact we could have emended our original guess by writing:

```
       1
       3
72)3 0 2 4    (Don't put
 - 2 1 6       the 1 over
     8 6      the 4.  We're
   - 7 2      still working
     1 4      with 302)
```

(2) But suppose you guess too high.

```
       5
72)3 0 2 4
 - 3 6 0
```

The clue now is that we're being called upon to subtract more than what we have. AND THIS CAN'T BE DONE WITH WHOLE NUMBERS.

Notice that we're trying to subtract 360 from 302. Don't confuse this with subtracting 360 from 3,024.

<u>*KEY POINT*</u>

AFTER EACH SUBTRACTION (AND BEFORE WE BRING DOWN THE NEXT DIGIT) THE ANSWER(DIFFERENCE) HAS TO BE AT LEAST 0 BUT LESS THAN THE DIVISOR.

We can't subtract more than what we have; and if the difference is more than the divisor we could have subtracted a greater multiple of the divisor.

And so we come to the end of our treatment of the arithmetic of whole numbers. In a sense subtraction, multiplication, and division are forms of addition--but division plays a very important role in the mathematics of the real world. Many times we are much more interested in how <u>*fast*</u> a quantity is changing than we are in the change itself. When we ask "How fast?" we're talking about <u>*rates*</u>. It is usually easy to recoginize a rate. It is a phrase consisting of two nouns, separ-

For example, if you spot a police car while you're driving you tend to glance at your speedometer--not at a road map. At that time you are more interested in how fast you're going than in where you are.

ated by the word *"per."* Examples of rates are: "miles per hour", "dollars per person", "miles per gallon", "dollars per pound", or even "apples per lawyer".

We get rates by dividing one quantity by another. For example if you drive at a constant speed and go 100 miles in 2 hours, your speed was 50 miles per hour. That is, we take the distance we went (100 miles) and divide it by the time it took to travel that distance (2 hours). The result is

100 miles ÷ 2 hours.

We divide the 100 by 2 to get 50; *but we also divide "miles" by "hours" to get "miles per hour".* Because of how important rates are, we often have to divide one whole number by another. *In most cases the quotient of two whole numbers is* not *a whole number. This forces us to invent a new--and broader--kind of number.* The quotient of two whole numbers is called a *rational number.* In the next several modules we revisit the arithmetic of whole numbers as we begin our study of the rational numbers.

The word "rational" comes from the word "ratio", which in turn suggests "rate". When we deal with ratios we divide one number by another.